GENERALISED PARTIALLY BALANCED DESIGN

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1. Introduction

In varietal trials if the number of varieties is large, the ordinary Randomized Block or Latin Square designs are not at all suitable as the efficiency of varietal comparisons become very much reduced. In order to overcome the drawback, a series of designs known as Quasi-factorial designs was first introduced by Yates (1936). As such designs could accommodate only those numbers of varieties which are only perfect squares or cubes, the utility of such designs proved to be limited. Afterwards, Yates (1936) introduced another design known as Balanced Incomplete Block Design which could accommodate more types of varietal trials. But this design has the defect that large number of replications for each of the varieties is required for its application. This evidently demands more resources. This difficulty was removed to some extent by Bose and Nair (1939) who introduced another series of designs known as Partially Balanced Incomplete Block Designs. Such designs can be obtained with smaller number of replications of each of the varieties. Actually, all the Quasifactorial designs together with many more available in literature are particular cases of such design.

All these designs were evolved having an eye to agricultural experiments only and as such suffered from some limitations. One essential requirement of varietal trials is that the block size should be small and no replication of treatment within block is permissible. But in experiments involving single plant progenies with variable amount of seeds or small number of treatments with litter mates as experimental units, it becomes desirable to adopt super-complete block designs, *i.e.*, designs allowing replication within groups (blocks) in order to avoid wastage of seed or animals. For example, if there be several experimental treatments to be applied to animals available in litters of size greater than the number of treatments, the usual practice is to discard the extra animals, taking as many animals from each litter as there are treatments. The experiment, on the other hand, could be suitably designed so as to utilize all the animals in the litters and this would ensure not only an increase in precision but also save wastage of animals resources. Such designs become objectionable for agricultural experimentation because it increases block size and thereby some precision is lost. But in experiments with animals as experimental units, such objection does not arise as the variation within litter (block) does not most likely depend on the size of the litter. The same consideration calls for designs by means of which litters of different sizes can be utilized in the same experiment which in field experiments means designs with unequal block sizes.

Taking into account all these facts, a Generalised Partially Balanced Design has been defined with provision of replication within blocks, if so necessary. The R.B.D., P.B.I.B.D., reinforced P.B.I.B. (*i.e.*, usual partially B.I.B.D. with some extra treatments which are present in each of the blocks), etc., come out as particular cases of this design. The method of intra-block analysis of this design together with the expressions of variances of treatment differences has also been worked out. The results obtained can be taken to be a generalization of all types of B.I.B. and P.B.I.B. designs available in literature.

The initiative to develop such design was taken by Das (1957) where he discussed the generalised balanced design. This design also suffers from the same defect that large number of blocks is required for balance and this calls for the Generalised Partially Balanced designs along with other generalised incomplete block designs.

2. Definition of the Design

If there be b blocks (a block being defined as a group of units like plots or animals having some common feature) and $v' + v$ treatments such that the i-th treatment in the j-th block occurs n_{ij} times, where n_{ij} is constant for all $j\,(= 1, 2, \ldots, b)$ being equal to $n \geqslant 0$ for the first set of v' treatments, while for the second set of v treatments it takes two types of values in these blocks, *viz.*, $S + p$ and S ($S \geqslant 0$, $p \geqslant 0$) such that the cells (a cell being defined by the combination of a treatment and block) receiving the frequency $S + p$ give rise to a Partially Balanced Incomplete Block Design with v-treatments in b blocks. Another b' block may be added to these b blocks such that the frequencies are the same within the same block, though they may differ from block to block.

Such a design involving $t\,(= v' + v)$ treatments and $b + b'$ blocks has been called a "Generalised Partially Balanced Design". Taking the parameters of the P.B.I.B. Design with s associates as b, v, r, k,

λ_j, n_j ($j = 1, 2, \ldots, s$) the cell frequencies, marginal totals, *i.e.*, block sizes and replications have been presented below in a tabular form.

TABLE OF FREQUENCIES

Blocks \ Treatments	1st Set 1	2	3	$\cdots v'$	2nd Set $v'+1$	$v'+2$	$\cdots v'+v$	Total Frequency
1 ..	n	n	n	$\cdots n$	$S+p$	S	$\cdots S$	$K = nv' + vS + kp$
2 ..	n	n	n	$\cdots n$	S	S	$\cdots S+p$	K
3 ..	n	n	n	$\cdots n$	S	$S+p$	$\cdots S$	.
. ..	. . .				. . .			.
. ..	. . .				. . .			.
. ..	. . .				. . .			.
b ..	n	n	n	$\cdots n$	S	$S+p$	$\cdots S+p$	K
$b+1$ ..	f_1	f_1		$\cdots f_1$	f_1	f_1	$\cdots f_1$	tf_1
$b+2$ ..	f_2	f_2		$\cdots f_2$	f_2	f_2	$\cdots f_2$	tf_2
. ..	. . .				. . .			.
. ..	. . .				. . .			.
. ..	. . .				. . .			.
. ..	. . .				. . .			.
$b+b'$ ..	$f_{b'}$	$f_{b'}$		$\cdots f_{b'}$	$f_{b'}$	$f_{b'}$	$\cdots f_{b'}$	$tf_{b'}$
Totals ..	R_1	R_1		$\cdots R_1$	R_2	R_2	$\cdots R_2$	$v'R_1 + vR_2 = bk + t\Sigma f$

where

$$R_1 = bn + \Sigma f, \quad R_2 = bS + rp + \Sigma f = R + \Sigma f \text{ (say)}$$

(i) When $n = S = 1$ and $f_i = 1$, $p = 0$ it becomes the ordinary Randomised Block Design.

(ii) When $p = 1$ and $f = n = S = 0$ it becomes the usual Partially Balanced Incomplete Block Design defined by Bose and Nair.

(iii) When $n = p = 1$ and $f_i = S = 0$ it becomes reinforced P.B.I.B. Design as defined in the first chapter.

In addition, various super-complete block designs which will be useful for experiments with litter mates can also be obtained from the General designs by giving suitable values to the defining parameters.

3. Method of Analysis

If y_{ijk} denotes the k-th observation from the i-th treatment in the j-th block, then taking the model $y_{ijk} = \mu + t_i + b_j + e_{ijk}$ where μ is constant, t_i and b_j are respectively the effects of the i-th treatment and jth block and e_{ijk}, a random variable with zero mean and variance σ^2, the normal equations for estimating the treatment effects after eliminating the block effects can be obtained by following Das (1953) with the help of the tables in next page.

Let m stands for a treatment in the first set and i for one in the second set. Then we have, following the notations used by Das (1953), *viz.*,

$$P_{\alpha\beta} = -\sum_j \frac{n_{\alpha j}(n_{\beta j} - n)}{n_{\cdot j}} \quad \text{when} \quad \alpha \neq \beta \cdot \begin{pmatrix} \alpha = 1, 2, \ldots, t \\ \beta = 2, 3, \ldots, t \end{pmatrix}$$

$$= R_\alpha - \sum_j \frac{n_{\alpha j}(n_{\alpha j} - n)}{n_{\cdot j}} \quad \text{when} \quad \alpha = \beta.$$

$n_{\cdot j}$ being the size of the j-th block

$$P_{mm} = R_1$$

$$P_{mi} = \frac{n(R - bn)}{K} \quad \text{where} \quad R = bS + rp$$

$P_{mm'} = 0$ where m' denotes another treatment in the first set.

$$P_{ii} = R_2 - \frac{r(S+p)(S+p-n) + (b-r)S(S-n)}{K} = R_2 - \frac{P}{K}.$$

$$\text{where} \quad P = r(S+p)(S+p-n) + (b-r)S(S-n)$$
$$= bS(S-n) + pr(2S-n+p)$$

$$P_{im} = 0$$

$$P_{ii'} = \frac{1}{K}\{\lambda_j(S+p)(S+p-n) + (r-\lambda_j)(S+p)(S-n)$$
$$+ (r-\lambda_j)S(S+p-n) + (b-2r+\lambda_j)(S-n)S\}$$
$$= \frac{1}{K}\{bS(S-n) + pr(2S-n) + p^2\lambda_j\} = \frac{L_j}{K} \quad \text{(say)},$$

Blocks	Table of frequencies									Totals	Table of differences (j-th column − 1st column) $j=(2 \cdots t)$							
	1st set of treatments					2nd set of treatments												
	1	2	3	$\cdots$	v'	$v'+1$	$v'+2$	$\cdots$	$v'+v$		2	3	$\cdots$	v'	$v'+1$	$v'+2$	$\cdots$	$v'+v$
1	n	n	n	$\cdots$	n	$S+p$	S	$\cdots$	S	K	0	0	$\cdots$	0	$S+p-n$	$S-n$	$\cdots$	$S-n$
2	n	n	n	$\cdots$	n	S	S	$\cdots$	$S+p$	K	0	0	$\cdots$	0	$S-n$	$S-n$	$\cdots$	$S+p-n$
3	n	n	n	$\cdots$	n	S	$S+p$	$\cdots$	S	K	0	0	$\cdots$	0	$S-n$	$S+p-n$	$\cdots$	$S-n$
·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
b	n	n	n	$\cdots$	n	S	$S+p$	$\cdots$	$S+p$	K	0	0	$\cdots$	0	$S-n$	$S+p-n$	$\cdots$	$S+p-n$
$b+1$	f_1	f_1	f_1	$\cdots$	f_1	f_1	f_1	$\cdots$	f_1	tf_1	0	0	$\cdots$	0	0	0	$\cdots$	0
$b+2$	f_2	f_2	f_2	$\cdots$	f_2	f_2	f_2	$\cdots$	f_2	tf_2	0	0	$\cdots$	0	0	0	$\cdots$	0
·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
$b+b$	$f_{b'}$	$f_{b'}$	$f_{b'}$	$\cdots$	$f_{b'}$	$f_{b'}$	$f_{b'}$	$\cdots$	$f_{b'}$	$tf_{b'}$	0	0	$\cdots$	0	0	0	$\cdots$	0

where i, i' denote two treatments which are j-th associates in the second set. As $P_{ii'}$ involves only four different types of products, *viz.*,

(i) $(S + p)(S + p - n)$

(ii) $(S + p)(S - n)$

(iii) $S(S + p - n)$

(iv) $S(S - n)$

and that the frequencies with which the products appear are respectively equal to λ_j, $r - \lambda_j$, $r - \lambda_j$ and $b - 2r + \lambda_j$, the expression for $P_{ii'}$ can be obtained as above.

Now, the normal equations for the treatmets in the first set come out as

$$R_1 t_m - \frac{n(R - bn)}{K} \sum t_i = Q_m \tag{1}$$

$$\begin{pmatrix} i = v' + 1, v' + 2, \ldots, v' + v \\ m = 1, 2, 3, \ldots, v' \end{pmatrix}$$

where Q_m is the adjusted total of the m-th treatment as defined by Das (1953).

The normal equations for the second set are:—

$$\left(R_2 - \frac{P}{K}\right) t_{v'+1} - \frac{L_1}{K} \Sigma_1 t_{v'+1} - \frac{L_2}{K} \Sigma_2 t_{v'+1} - \ldots - \frac{L_s}{K} \Sigma_s t_{v'+1} = Q_{v'+1}$$

$$\left(R_2 - \frac{P}{K}\right) t_{v'+2} - \frac{L_1}{K} \Sigma_1 t_{v'+2} - \frac{L_2}{K} \Sigma_2 t_{v'+2} - \ldots - \frac{L_s}{K} \Sigma_s t_{v'+2} = Q_{v'+2}$$

....

....

$$\left(R_2 - \frac{P}{K}\right) t_{v'+v} - \frac{L_1}{K} \Sigma_1 t_{v'+v} - \frac{L_2}{K} \Sigma_2 t_{v'+v} - \ldots - \frac{L_s}{K} \Sigma_s t_{v'+v} = Q_{v'+v}.$$

In general, the above equations can be written as

$$\left(R_2 - \frac{P}{K}\right) t_i - \sum_{j=1}^{s} \frac{L_j}{K} (\Sigma_j t_i) = Q_i \tag{2}$$

$$(i = v' + 1, v' + 2, \ldots, v' + v)$$

Where $\Sigma_j t_i$ denotes the sum of those treatments which are the j-th associates to the i-th treatment; and Q_i is the adjusted total of the i-th treatment as defined by Das (1953).

It is seen that the solution of (2), *i.e.*, the second set of equations is independent of that of (1), *i.e.*, the first set of equations, the solution for which can be obtained easily once the solution of the second set is available.

Equations (2) can be written as:

$$(R_2K - P) t_i - \sum_{j=1}^{s} L_j (\Sigma_j t_i) = KQ_i \tag{3}$$

Adding over i (v in number) we get

$$(R_2K - P) \Sigma t_i - n_1L_1 \Sigma t_i - n_2L_2 \Sigma t_i - \ldots - n_sL_s \Sigma t_i = K \Sigma Q_i$$

or

$$\left(R_2K - P - \sum_{j=1}^{s} n_jL_j\right) \Sigma t_i = K \Sigma Q_i$$

$$\therefore \quad \sum t_i = \frac{K \Sigma Q_i}{\left(R_2K - P - \sum_j n_jL_j\right)} \tag{4}$$

But

$$P + \sum_j n_jL_j = R(S - n) + rp(S + p) + R(S - n)(v - 1)$$

$$+ Rpk - pr(S + p)$$

$$= R(K - nt).$$

$$\therefore \quad R_2K - P - \sum_j n_jL_j = K(R + \Sigma f) - R(K - nt)$$

$$= Rnt + K \Sigma f.$$

Hence finally

$$\sum t_i = \frac{K \Sigma Q_i}{Rnt + K \Sigma f} \tag{5}$$

The solution of the first set of equations comes out as:

$$t_m = \frac{1}{R_1} \left\{ Q_m + \frac{n(R - bn) \Sigma Q_i}{Rnt + K \Sigma f} \right\}. \tag{6}$$

A scrutiny of the second set of normal equations shows that these have come in the same form as obtained by Bose and Nair in the ordinary P.B.I.B. design, the only difference being that $r(k-1)$ has been replaced by $R_2 K - P$ and the λ_j's by L_j's.

Adding equations (2) over those treatments which are the j-th associates to the i-th treatment we get

$$\left(R_2 - \frac{P}{K}\right) \sum_j t_i - \frac{L_1}{K} (p_{j1}{}^1 \Sigma_1 t_i + p_{j1}{}^2 \Sigma_2 t_i + \ldots + p_{j1}{}^s \Sigma_s t_i)$$

$$- \frac{L_2}{K} (p_{j2}{}^1 \Sigma_1 t_i + p_{j2}{}^2 \Sigma_2 t_i + \ldots + p_{j2}{}^s \Sigma_s t_i)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$- \frac{L_j}{K} (n_j t_i + p_{jj}{}^1 \Sigma_1 t_i + p_{jj}{}^2 \Sigma_2 t_i + \ldots + p_{jj}{}^s \Sigma_s t_i)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$- \frac{L_s}{K} (p_{js}{}^1 \Sigma_1 t_i + p_{js}{}^2 \Sigma_2 t_i + \ldots + p_{js}{}^s \Sigma_s t_i)$$

$$= \Sigma_j Q_i \tag{7}$$

Where $\Sigma_j Q_i$ is the total of those Q's which are j-th associate to Q_i. Equation (7) can be re-arranged in the form

$$\{R_2 K - P - (L_1 p_{j1}{}^j + L_2 p_{j2}{}^j + \ldots + L_s p_{js}{}^j)\} \Sigma_j t_i$$

$$- (L_1 p_{j1}{}^1 + L_2 p_{j2}{}^1 + \ldots + L_s p_{js}{}^1) \Sigma_1 t_i$$

$$- (L_1 p_{j1}{}^2 + L_2 p_{j2}{}^2 + \ldots + L_s p_{js}{}^2) \Sigma_2 t_i$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$- (L_1 p_{j1}{}^s + L_2 p_{j2}{}^s + \ldots + L_s p_{js}{}^s) \Sigma_s t_i$$

$$- L_j n_j t_i = K \Sigma_j Q_i \tag{8}$$

or

$$\left\{R_2K - P - \left(\sum_{j'=1}^{s} L_{j'}p_{jj'}{}^{j}\right)\right\} \Sigma_j\, t_i - \left(\sum_{j'} L_{j'}p_{jj'}{}^{1}\right) \Sigma_1\, t_i$$
$$- \left(\sum_{j'} L_{j'}p_{jj'}{}^{2}\right) \Sigma_2\, t_i \ldots \left(\sum_{j'} L_{j'}p_{jj'}{}^{s}\right) \Sigma_s\, t_i$$
$$- L_jn_jt_i = K\Sigma_j\, Q_i \tag{9}$$

It is seen that the coefficient of $\Sigma_k\, t_i$ (say) in the ordinary case, *viz.*, $\sum_{j'} \lambda_{j'}p_{jj'}{}^{k}$ has a definite meaning. This is actually the number of times the j-th associate of a treatment θ (say) occurs with another treatment ϕ where θ and ϕ are k-th associates. Denoting this number by δ_{kj} and the corresponding expressions in the general case, *viz.*, $\sum_{j'=1}^{s} L_{j'}p_{jj'}{}^{k}$ by Δ_{kj}, the equation (9) can be written as

$$- L_jn_jt_i - \Delta_{1j}\Sigma_1\, t_i - \Delta_{2j}\Sigma_2\, t_i \ldots + (R_2K - P - \Delta_{jj})\,\Sigma_jt_i$$
$$- \ldots - \Delta_{sj}\Sigma_s\, t_i = K\Sigma_j\, Q_i \tag{10}$$

Eliminating $\Sigma_s\, t_i$ with the help of the relation

$$\Sigma\, t_i = t_i + \Sigma_1\, t_i + \Sigma_2\, t_i + \ldots + \Sigma_s\, t_i \tag{11}$$

the equation (10) becomes

$$(\Delta_{sj} - L_jn_j)\, t_i + (\Delta_{sj} - \Delta_{1j})\,\Sigma_1\, t_i + (\Delta_{sj} - \Delta_{2j})\,\Sigma_2\, t_i + \ldots$$
$$+ (R_2K - P + \Delta_{sj} - \Delta_{jj})\,\Sigma_j\, t_i + \ldots$$
$$+ (\Delta_{sj} - \Delta_{s-1,\,j})\,\Sigma_{s-1}\, t_i$$
$$= K\Sigma_j\, Q_i + \Delta_{sj}\,\Sigma t_i. \tag{12}$$

As it is seen that

(i) $\Delta_{sj} - L_jn_j = p^2\,(\delta_{sj} - \lambda_jn_j)$

(ii) $\Delta_{sj} - \Delta_{1j} = p^2\,(\delta_{sj} - \delta_{1j})$

(iii) $R_2K - P + \Delta_{sj} - \Delta_{jj} = R_2K - p^2r + p^2\,(\delta_{sj} - \delta_{jj})$

it is better to experess the equations in terms of the δ_{kj}'s which are directly obtainable. The final set of equations corresponding (12) can thus be written as

$$\{p^2\,(\delta_{sj} - \lambda_jn_j)\}\, t_i + \{p^2\,(\delta_{sj} - \delta_{1j})\}\,\Sigma_1\, t_i + \{p^2\,(\delta_{sj} - \delta_{2j})\}\,\Sigma_2\, t_i$$
$$+ \ldots + \{R_2K - p^2r + p^2\,(\delta_{sj} - \delta_{jj})\}\,\Sigma_j\, t_i$$
$$+ \ldots + \{p^2\,(\delta_{sj} - \delta_{s-1,\,j})\}\,\Sigma_{s-1}\, t_i$$
$$= K\Sigma_j\, Q_i + \Delta_{sj}\,\Sigma\, t_i \tag{13}$$

$$(j = 1, 2, \ldots s - 1)$$

From (13) we get $(s-1)$ equations involving the 's' unknowns, *viz.*,

$$t_i, \Sigma_1 t_i, \Sigma_2 t_i, \ldots, \Sigma_{s-1} t_i.$$

Eliminating $\Sigma_s t_i$ from equation (3) with the help of the relation (11) we get

$$\begin{aligned}(R_2K - P + L_s)\, t_i + (L_s - L_1)\, \Sigma_1 t_i + (L_s - L_2)\, \Sigma_2 t_i \\ + \ldots + (L_s - L_{s-1})\, \Sigma_{s-1} t_i \\ = K Q_i + L_s \Sigma t_i \end{aligned} \qquad (14)$$

As

$$L_s - L_j = p^2 (\lambda_s - \lambda_j)$$

and

$$(R_2K - P + L_s) = R_2K - p^2 (r - \lambda_s)$$

equation (14) can also be written in term of λ's, *i.e.*,

$$\begin{aligned}\{R_2K - p^2 (r - \lambda_s)\}\, t_i + p^2 (L_s - \lambda_1)\, \Sigma_1 t_i + p^2 (\lambda_s - \lambda_2)\, \Sigma_2 t_i \\ + \ldots + p^2 (\lambda_s - \lambda_{s-1})\, \Sigma_{s-1} t_i \\ = K Q_i + L_s \Sigma t_i \end{aligned} \qquad (15)$$

Thus from (13) along with (15) we have got 's' equations involving 's' unknowns. A general solution of t_i can now be obtained from these equations. Actually the solution is of the form

$$t_i = C_0 M_i + \sum_{j=1}^{s-1} C_j M_{ij}$$

where

$$\begin{aligned} M_i &= K Q_i + L_s \Sigma t_i \\ M_{ij} &= K \Sigma_j Q_i + \Delta_{sj} \Sigma t_i. \end{aligned}$$

It must be mentioned here that the normal equations can be written more conveniently in terms of δ_{kj}'s than in terms of A's, B's, etc. as has been suggested by Bose and Nair, since the δ_{kj}'s can be treated as a further set of parameters of the design, and hence can be obtained directly.

Once the solution of the treatment effects $(t_1, t_2, \ldots, t_{v'+v}$ is obtained, the treatment sum of squares can be calculated as usual form ΣtQ and the sum of squares due to all the fitted constants from $\sum_j B_j^2/n_{.j} + \Sigma tQ$ where

B_j: total for the j-th block

n_j: number of observations in the j-th block.

$$(j = 1, 2, \ldots, b + b')$$

By subtracting the sum of squares due to all the fitted constants as given above from $\Sigma\, T_{ij}^2/n_{ij}$ where T_{ij} is the total of the i-th treatment as obtained from the j-th block, the interaction sum of squares can be obtained. The within block pure error sum of squares can be obtained as usual from the within cell sum of squares.

4. Particular Cases

(i) *Case of two associates.*—

When there are two associates, *i.e.*, $s = 2$ the normal equations for estimating the treatments in the second set can easily be written from the general equations given in (13) and (15).

They turn out to be

$$\left.\begin{aligned} &\{p^2(\delta_{21}-\lambda_1 n_1)\}\, t_i + \{R_2K - p^2 r + p^2(\delta_{21}-\delta_{11})\}\, \Sigma_1\, t_i \\ &\qquad = K\,\Sigma_1\, Q_i + \Delta_{21}\,\Sigma t_i \\ \text{and}\quad & \\ &\{R_2K - p^2(r-\lambda_2)\}\, t_i + p^2(\lambda_2 - \lambda_1)\,\Sigma_1\, t_i \\ &\qquad = KQ_i + L_2\,\Sigma\, t_i \end{aligned}\right\} \tag{16}$$

Utilizing the relations

$$p_{11}^1 + p_{12}^1 = n_1 - 1; \qquad p_{11}^2 + p_{12}^2 = n_1$$

$$p_{21}^2 + p_{22}^2 = n_2 - 1; \qquad p_{21}^1 + p_{22}^1 = n_2$$

and taking the equations in term of Δ's and P's we find

$$\begin{aligned} \Delta_{21} - L_1 n_1 &= L_1 p_{11}^2 + L_2 p_{12}^2 - n_1 L_1 \\ &= (L_2 - L_1)\, p_{12}^2 \qquad \left(\because\ \Delta_{kj} = \sum_{j'} L_{j'} p_{jj'}^k\right) \end{aligned}$$

and

$$\Delta_{21} - \Delta_{11} = (L_2 - L_1)(p_{11}^1 - p_{11}^2) + L_2.$$

Hence, using the notations of Bose and Nair (1939) we find that in the general case having two associations, the equations (16) can be written in the form

$$\left.\begin{aligned} A_{12} t_i + B_{12}\,\Sigma_1\, t_i &= M_i \\ A_{22} t_i + B_{22}\,\Sigma_1\, t_i &= M_{i1} \end{aligned}\right\} \tag{17}$$

where

$$A_{12} = (R_2K - P) + L_2 = R_2K - p^2(r - \lambda_2)$$

$$A_{22} = (L_2 - L_1)\, p_{12}^2 = p^2(\delta_{21} - \lambda_1 n_1)$$

$$B_{12} = (L_2 - L_1) = p^2 (\lambda_2 - \lambda_1)$$

$$B_{22} = (R_2K - P) + L_2 + (L_2 - L_1)(p_{11}^1 - p_{11}^2)$$

$$= R_2K - p^2r + p^2 (\delta_{21} - \delta_{11})$$

$$M_i = KQ_i + L_2 \Sigma t_i$$

$$M_{i1} = K\Sigma_1 Q_i + \Delta_{21} \Sigma t_i$$

Thus, the solutions for t_i is of the same form as given by Bose and Nair but for the new meaning of A_{12}, A_{22}, B_{12} and B_{22} and also that KQ_i and $K\Sigma_1 Q_i$ are to be replaced by M_i and M_{i1} respectively, Σt_i being obtained from (5).

$$\therefore \quad t_i = C_0 M_i + C_1 M_{i1} \tag{18}$$

where

$$C_0 = \frac{B_{22}}{\Delta}, \quad C_1 = -\frac{B_{12}}{\Delta}$$

$$\Delta = A_{12} B_{22} - A_{22} B_{12}.$$

The general solution for t_m, *i.e.*, the treatments in the first set has already been presented in (6).

Following the procedure adopted by Das (1953) the variances of the estimates of $t_m - t_{m'}$, $t_m - t_i$ and $t_i - t_i'$ (when $i\,i'$ are both first associates and second associates) have been obtained as below:—

$$V(t_m - t_{m'}) = \sigma^2 \left(\frac{1}{R_1} + \frac{1}{R_1}\right) = \frac{2\sigma^2}{R_1} \tag{19}$$

(m, m' are treatments of the first set).

$$V(t_m - t_i) = \sigma^2 \left[\left\{\frac{1}{R_1} - \frac{n(R - bn)}{R_1 l}\right\} + \left\{C_0\left(K + L_2 \frac{K}{l}\right) + C_1 \Delta_{21} \frac{K}{l}\right\}\right]$$

$$= \sigma^2 \left[\frac{1}{R_1} - \frac{n(R - bn)}{R_1 l} + C_0 K + \frac{K}{l}(C_0 L_2 + C_1 \Delta_{21})\right]. \tag{20}$$

where $l = Rnt + K\Sigma f$

(m, i are the treatments of first and second set respectively)

$$V(t_i - t_{i'}) = 2\sigma^2 K (C_0 - C_1) \qquad (21)$$

($i\,i'$ are treatments of second set and are first associates)

$$= 2\sigma^2 C_0 K \qquad (22)$$

(when $i\,i'$ are the second associates)

The expression for average variance comes out as

$$\frac{2\sigma^2 K}{v-1}\{n_1 C_0 - n_1 C_1 + n_2 C_0\}$$

$$= \frac{2\sigma^2 K}{v-1}\{(v-1)\, C_0 - n_1 C_1\}$$

$$= \frac{2\sigma^2 K}{(v-1)\,\Delta}\{(v-1)\, B_{22} + n_1 B_{12}\} \qquad (23)$$

(*ii*) *Case of three associates*—

The normal equations for the second set of treatments in the case of three associates come out as

$$\left.\begin{aligned}
&\{R_2 K - p\,(r-\lambda_3)\}\, t_i + p^2(\lambda_3 - \lambda_1)\,\Sigma_1\, t_i \\
&\qquad + p^2(\lambda_3 - \lambda_2)\,\Sigma_2\, t_i = K Q_i + L_3\,\Sigma\, t_i \\
&\{p^2(\delta_{31} - \lambda_1 n_1)\}\, t_i + \{R_2 K - p^2 r + p^2(\delta_{31} - \delta_{11})\}\,\Sigma_1\, t_i \\
&\qquad + p^2(\delta_{31} - \delta_{21})\,\Sigma_2\, t_i = K\,\Sigma_1\, Q_i + \Delta_{31}\,\Sigma\, t_i \\
&\{p^2(\delta_{32} - \lambda_2 n_2)\}\, t_i + p^2(\delta_{32} - \delta_{12})\,\Sigma_1\, t_i + \{R_2 K - p^2 r \\
&\qquad + p^2(\delta_{32} - \delta_{22})\}\,\Sigma_2\, t_i = K\,\Sigma_2\, Q_i + \Delta_{32}\,\Sigma\, t_i
\end{aligned}\right\} \qquad (24)$$

The above equations can be expressed as Δ's and P's from the equations (12) and (14).

With the help of the relations

$$\Delta_{kj} = \sum_{j'} L_{j'} p_{jj'}{}^{k}$$

$$\sum_{K} p_{jk}{}^{i} = n_j \qquad (i \neq j)$$

$$= n_j - 1 \qquad (i = j)$$

the normal equations can also be arranged as

$$\left.\begin{aligned}
A_{13} t_i + B_{13}\,\Sigma_1\, t_i + C_{13}\,\Sigma_2\, t_i &= M_i \\
A_{23} t_i + B_{23}\,\Sigma_1\, t_i + C_{23}\,\Sigma_2\, t_i &= M_{i1} \\
A_{33} t_i + B_{33}\,\Sigma_1\, t_i + C_{33}\,\Sigma_2\, t_i &= M_{i2}
\end{aligned}\right\} \qquad (25)$$

where

$$A_{13} = R_2K - P + L_3$$

$$B_{13} = L_3 - L_1$$

$$C_{13} = L_3 - L_2$$

$$A_{23} = (L_3 - L_1)(n_1 - p_{11}^3) - (L_3 - L_2)\, p_{12}^3$$

$$B_{23} = (R_2K - P + L_3) + (L_3 - L_1)(p_{11}^1 - p_{11}^3) + (L_3 - L_2)(p_{12}^1 - p_{12}^3)$$

$$C_{23} = (L_3 - L_1)(p_{11}^2 - p_{11}^3) + (L_3 - L_2)(p_{12}^2 - p_{12}^3)$$

$$A_{33} = (L_3 - L_2)(n_2 - p_{22}^3) - (L_3 - L_1)\, p_{12}^3$$

$$B_{33} = (L_3 - L_1)(p_{12}^1 - p_{12}^3) + (L_3 - L_2)(p_{22}^1 - p_{22}^3)$$

$$C_{33} = (R_2K - P + L_3) + (L_3 - L_1)(p_{12}^2 - p_{12}^3) + (L_3 - L_2)(p_{22}^2 - p_{22}^3)$$

and

$$M_i = KQ_i + L_3 \Sigma t_i$$

$$M_{i1} = K \Sigma_1 Q_i + \Delta_{31} \Sigma t_i$$

$$M_{i2} = K \Sigma_2 Q_i + \Delta_{32} \Sigma t_i.$$

Hence

$$t_i = \begin{vmatrix} M_i & B_{13} & C_{13} \\ M_{i1} & B_{23} & C_{23} \\ M_{i2} & B_{33} & C_{33} \end{vmatrix} \div \begin{vmatrix} A_{13} & B_{13} & C_{13} \\ A_{23} & B_{23} & C_{23} \\ A_{33} & B_{33} & C_{33} \end{vmatrix} \qquad (26)$$

Now it is seen that the equations have been brought in the form given by Bose and Nair but for the new meanings of the coefficients, $A_{13}, B_{13}, \ldots, C_{33}$ as stated above.

The variance of $(t_m - t_{m'})$ remains $2\sigma^2/R_1$. Variances of $(t_i - t_{i'})$ are functions of the coefficients A_{13}, B_{13}, etc., and as such can be detained without difficulty following Bose and Nair.

Variance of $(t_m - t_i)$ will be in this case

$$\sigma^2 \left[\frac{1}{R_1} - \frac{n\,(R - bn)}{R_1 l} + C_0 K + \frac{K}{l}(C_0 L_3 + C_1 \Delta_{31} + C_2 \Delta_{32}) \right] \qquad (27)$$

In fact, the variance of $(t_m - t_i)$ in the general case of 's' associates come out as

$$\sigma^2 \left[\frac{1}{R_1} - \frac{n(R - bn)}{R_1 l} + C_0 K + \frac{K}{l} (C_0 L_s + \sum_{j=1}^{s-1} C_j \Delta_{sj}) \right] \qquad (28)$$

where C_j is the coefficient of M_{ij} in the solution of t_i.

and

$$M_{ij} = K \Sigma_j Q_i + \Delta_{sj} \Sigma t_i.$$

As has been mentioned earlier giving particular value to S, n, p, f_i, v', etc., various useful designs can be obtained from the general design. The analytical results of them also can be obtained by simply substituting for these parameters in the various results. As for example, in the case of reinforced P.B.I.B. design mentioned earlier, we get

$$R_2 = r, \; K = v' + k, \; P = 0, \; L_j = -(r - \lambda_j)$$

$$R_2 K - P + L_s = r(v' + k) - r + \lambda_j$$

$$R_2 K - p^2 r + p^2 (\delta_{sj} - \delta_{jj}) = r(k + v') - r + \delta_{sj} - \delta_{jj}.$$

It is seen that the solution of the treatments in the second set can be obtained from the corresponding expressions in the ordinary P.B.I.B. design simply by replacing k by $k + v'$. If we put $\lambda_j = \lambda$ for all j's in the various results, we get the corresponding results for the B.I.B. designs.

5. An Example

The method of analysis has been illustrated through the following example. The data analysed were artificially constructed from the Uniformity trial data on Malvi Cotton reported by Hutchinson and Panse (1935) such that they formed a progeny row trials with 17 plants in 20 blocks. Taking that the seeds available were sufficient for 20 replications for each of five plants while for each of the remaining 12 plants, there were seeds sufficient for 45 replications, one of the suitable design is a super complete Partially Balanced Incomplete Block Design built out of the parameters.

$$v = 12, b = 20, v' = 5, n = 1, S = 2, p = 1, r = 5,$$

$$k = 3, n_1 = 5, \; \lambda_1 = 2, n_2 = 5, \lambda_2 = 0, n_3 = 1, \lambda_3 = 0$$

$$p_{ij}^1 = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; p_{ij}^2 = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; p_{ij}^3 = \begin{pmatrix} 0 & 5 & 0 \\ 5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The block contents of the originating design and other parameters necessary for the estimation of the treatment effects are shown below:—

Denoting the 12 treatments forming the P.B.I.B. design by 6, 7, 8, ..., 17 the block contents are:

(6, 7, 8), (6, 8, 9), (6, 9, 10), (6, 10, 11), (6, 11, 7) (7, 8, 15), (8, 9, 16), (9, 10, 12) (10, 11, 13), (11, 7, 14), (7, 14, 15), (8, 15, 16), (9, 16, 12), (10, 12, 13), (11, 13, 14), (12, 13, 17), (13, 14, 17), (14, 15, 17), (15, 16, 17), (16, 12, 17).

Parameters necessary are:

$$R_2 = 45 = \mathrm{R},\ R_1 = 20,\ K = 32,\ P = 60$$

$$L_1 = 57,\ L_2 = 55,\ L_3 = 55$$

$$\Delta_{31} = 275,\ \Delta_{32} = 285.$$

$$\Sigma t_i = \frac{K \Sigma Q_i}{Rnt} = 15 \cdot 0314 \qquad (i = 6, 7, \ldots 17)$$

The normal equations for the second set of treatments $(t_6, t_7, \ldots t_{17})$ according to (25) are

$$1435\, t_i - 2\Sigma_1 t_i = 32 Q_i + 55\, \Sigma\, t_i\ (\equiv M_i)$$

$$-10\, t_i + 1431\, \Sigma_1 t_i - 4\, \Sigma_2 t_i = 32\, \Sigma_1 Q_i + 275\, \Sigma\, t_i\ (\equiv M_{i1})$$

$$10\, t_i + 6\, \Sigma_1 t_i + 1441\, \Sigma_2 t_i = 32\, \Sigma_2 Q_i + 285\, \Sigma\, t_i\ (\equiv M_{i2})$$

in which $Q_i, \Sigma_1 Q_i, \Sigma_2 Q_i$ and Σt_i are known. Solving the above equations we get

$$t_i = \frac{1}{2959077585}\,(2062095\, M_i + 2882\, M_{i1} + 8\, M_{i2}).$$

The values of t_m, *i.e.*, treatments in the first set are obtained from the equation (6).

The adjusted treatment totals (Q_i's) as obtained from the data are given below:—

Adjusted totals with	−76·47	−4·47	−152·47	−13·47	−112·47	116·47	−127·03	26·84	24·16
treatment Nos.	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)

Adjusted totals with	−4·72	−24·94	−8·84	165·72	−24·28	39·00	110·97	66·00
treatment Nos.	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)

The treatment effects have been obtained as:—

t_1	$-3\cdot2363$	t_{10}	$0\cdot4834$
t_2	$0\cdot3637$	t_{11}	$0\cdot0280$
t_3	$-7\cdot0363$	t_{12}	$0\cdot3942$
t_4	$-0\cdot0863$	t_{13}	$4\cdot2758$
t_5	$-5\cdot0363$	t_{14}	$0\cdot0424$
t_6	$3\cdot1741$	t_{15}	$1\cdot4515$
t_7	$-2\cdot2484$	t_{16}	$3\cdot0594$
t_8	$1\cdot1839$	t_{17}	$2\cdot0608$
t_9	$1\cdot1264$		

The checks that $\Sigma\, t = 0$ and also $t_i = K\Sigma\, Q_i/Rnt$ are satisfied. The S.S. due to the treatments obtained from $\Sigma\, tQ$ is $3833\cdot7394$.

Analysis of Variance Table

Sources of variation	d.f.	S.S.	M.S.
Between Blocks (unadjusted) ..	19	41310·02	..
Between Treatments (adjusted) ..	16	3833·74	239·61
Int (Blocks × Treatments)	304	77840·79	256·06
Error	300	167250·50	557·50
Total	639	290235·05	..

$$\text{Variance } (t_m - t_{m'}) = 0\cdot1\sigma^2$$

$$\text{Variance } (t_i - t_{i'}) = 0\cdot0445\sigma^2 \quad \text{(for first associates)}$$
$$= 0\cdot0446\sigma^2 \quad \text{(for second associates)}$$
$$= 0\cdot0446\sigma^2 \quad \text{(for third associates)}$$

$$\text{Variance } (t_m - t_i) = 0\cdot0722\sigma^2.$$

6. Summary

A generalised partially balanced design has been defined. The randomised block designs, the partially balanced incomplete block designs with or without some extra treatments which are present in every block and different partially balanced super complete block designs, come out as particular cases of the general design. The design is particularly helpful for single plant progeny row trials with variable amount of seed available from the different progenies and for animal experiments involving smaller number of treatments with elimination of litter effect, as in Bio-assays and other animal husbandry experiments. The method of analysis, together with the expressions for finding the standard errors of treatment differences, has been presented. The different steps involved in the analysis of the design have been illustrated by means of an example.

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